

q -Newton Binomial: From Euler To Gauss

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Abstract

A counter-intuitive result of Gauss (formulae (1.6), (1.7) below) is made less mysterious by virtue of being generalized through the introduction of an additional parameter.

1 A formula of Gauss revisited

Consider the Newton binomial for a positive integer N :

$$(1-x)^N = \sum_{\ell=0}^N \binom{N}{\ell} (1-x)^\ell. \quad (1.1)$$

Substituting $x = 1$ into this formula, we get

$$\sum_{\ell=0}^N \binom{N}{\ell} (-1)^\ell = 0. \quad (1.2)$$

What happens with these two equalities in the q -mathematics framework? Newton's formula (1) becomes Euler's formula

$$(1-x)^N = (1-x)(1-qx)\dots(1-q^{N-1}x) = \sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (-x)^\ell q^{\binom{\ell}{2}}, \quad (1.3)$$

where $\begin{bmatrix} N \\ \ell \end{bmatrix} = \begin{bmatrix} N \\ \ell \end{bmatrix}_q$ are the Gaussian polynomials, or q -binomial coefficients:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} = \frac{[n]\dots[n-k+1]}{[k]!}, \quad k \in \mathbf{Z}_+, \quad (1.4)$$

$$[k]! = [k]_q! = [1][2]\dots[k], \quad [0]! = 1, \quad [n] = [n]_q = (1-q^n)(1-q).$$

Substituting $x = 1$ into the Euler formula (1.3), we find

$$\sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (-1)^\ell q^{\binom{\ell}{2}} = 0. \quad (1.5)$$

This does not look exactly as a q -analogue of formula (1.2).

How about the sum $\sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (-1)^\ell$?

The answer is quite surprising. Denote

$$s_{N|0} = (-1)^N \sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (-1)^\ell. \quad (1.6)$$

Gauss found that

$$s_{2m+1|0} = 0, \quad m \in \mathbf{Z}_+, \quad (1.7a)$$

$$s_{2m+2|0} = (1-q)(1-q^3)\dots(1-q^{2m+1}), \quad m \in \mathbf{Z}_+. \quad (1.7b)$$

These formulae are easy to prove, but they are nevertheless mystifying: there is no hint in the definition (1.6) that some sort of 2-periodicity is involved. In addition, formula (1.2) may claim the following sums as proper q -analogues:

$$s_{N|1} = (-1)^N \sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (-q)^\ell, \quad (1.8)$$

or even

$$s_{N|r} = (-1)^N \sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (-q^r)^\ell. \quad (1.9)$$

Indeed, we shall verify later on that

$$s_{2m+1|1} = -(1-q^{2m+1})s_{2m|0} = -\prod_{t=0}^m (1-q^{2t+1}), \quad (1.10a)$$

$$s_{2m|1} = s_{2m|0} = \prod_{t=1}^m (1-q^{2t-1}). \quad (1.10b)$$

Similar but more complex formulae can be derived for other values of r , not just for $r = 0$ and $r = 1$. We shall abstain from such derivations, as they are superseded by the general formulae (1.12) below.

What seems to be happening here is that the functions

$$S_N(x) = (-1)^N \sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (-x)^\ell \quad (1.11)$$

possess some interesting properties worthy of attention; and once the decision to pay attention has been made, one quickly conjectures the formulae

$$\begin{aligned}
S_{2m+1}(x) &= - \sum_{\ell=0}^{2m+1} \begin{bmatrix} 2m+1 \\ \ell \end{bmatrix} (-x)^\ell \\
&= \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (x \dot{-} 1)^{2m+1-2k} (q^{2m+1}; q^{-2})_k,
\end{aligned} \tag{1.12a}$$

$$\begin{aligned}
S_{2m+2}(x) &= \sum_{\ell=0}^{2m+2} \begin{bmatrix} 2m+2 \\ \ell \end{bmatrix} (-x)^\ell \\
&= \sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} (x \dot{-} 1)^{2m+2-2k} (q^{2m+1}; q^{-2})_k.
\end{aligned} \tag{1.12b}$$

The additional notations employed above are to be understood as

$$(u \dot{+} v)^\ell = \prod_{k=0}^{\ell-1} (u + q^k v), \quad \ell \in \mathbf{N}; \quad (u \dot{+} v)^0 = 1, \tag{1.13}$$

and

$$(a; Q)_\ell = \prod_{k=0}^{\ell-1} (1 - Q^k a), \quad \ell \in \mathbf{N}; \quad (a; Q)_0 = 1. \tag{1.14}$$

If we define

$$\epsilon(N) = \begin{cases} 1, & N \text{ is even} \\ 0, & N \text{ is odd} \end{cases} = \left\lfloor \frac{N+2}{2} \right\rfloor - \left\lfloor \frac{N+1}{2} \right\rfloor, \tag{1.15}$$

then formulae (1.12) can be rewritten as

$$\begin{aligned}
S_N(x) &= (-1)^N \sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (-x)^\ell \\
&= \sum_{k=0}^{\lfloor N/2 \rfloor} \begin{bmatrix} \lfloor N/2 \rfloor \\ k \end{bmatrix}_{q^2} (x \dot{-} 1)^{N-2k} (q^{N-\epsilon(N)}; q^{-2})_k.
\end{aligned} \tag{1.16}$$

Substituting $x = 1$ into formulae (1.12) we recover Gauss' formulae (1.7).

Let us now prove formulae (1.12). Denote the RHS of formulae (1.16) by $\tilde{S}_N(x)$. To show that

$$S_N(x) = \tilde{S}_N(x), \tag{1.17}$$

we shall verify, first, that

$$\frac{dS_N}{d_q x} = [N] S_{N-1}, \tag{1.18a}$$

$$\frac{d\tilde{S}_N}{d_q x} = [N] \tilde{S}_{N-1}(x), \tag{1.18b}$$

and second, that

$$S_N(1) = \tilde{S}_N(1); \quad (1.19)$$

here

$$\frac{df(x)}{d_q x} = \frac{f(qx) - f(x)}{qx - x} \quad (1.20)$$

is the q -derivative. Since $S_1 = \tilde{S}_1 = x - 1$, these verifications would suffice.

We start with formula (1.18a). We have:

$$\begin{aligned} \frac{dS_N}{d_q x} &= \frac{d}{d_q x} \left((-1)^N \sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (-1)^\ell x^\ell \right) = (-1)^N \sum_{\ell=1}^N \begin{bmatrix} N \\ \ell \end{bmatrix} [\ell] (-1)^\ell x^{\ell-1} \quad [\text{by (1.22)}] \\ &= (-1)^N [N] \sum_{\ell=1}^N \begin{bmatrix} N-1 \\ \ell-1 \end{bmatrix} (-x)^{\ell-1} (-1) = [N] (-1)^{N-1} \sum_{\ell=0}^{N-1} \begin{bmatrix} N-1 \\ \ell \end{bmatrix} (-x)^\ell \\ &= [N] S_{N-1}, \end{aligned} \quad (1.21)$$

where we used the obvious formula

$$\begin{bmatrix} w \\ \ell \end{bmatrix} [\ell] = [w] \begin{bmatrix} w-1 \\ \ell-1 \end{bmatrix}. \quad (1.22)$$

Next, formula (1.18b), which we shall check separately for odd and even N , making use of the easy verifiable relation

$$\frac{d(x \dot{+} v)^\alpha}{d_q x} = [\alpha] (x \dot{+} v)^{\alpha-1}. \quad (1.23)$$

So, for N odd, we have

$$\frac{d\tilde{S}_1}{d_q x} = \frac{d}{d_q x} (x - 1) = 1 = \tilde{S}_0, \quad (1.24)$$

and then

$$\begin{aligned} \frac{d\tilde{S}_{2m+3}}{d_q x} &= \sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} [2m+3-2k] (x \dot{-} 1)^{2m+2-2k} (q^{2m+3}; q^{-2})_k \\ &= [2m+3] \sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} (x \dot{-} 1)^{2m+2-2k} (q^{2m+1}; q^{-2})_k = [2m+3] \tilde{S}_{2m+2}, \end{aligned}$$

because

$$[2m+3-2k] (q^{2m+3}; q^{-2})_k = [2m+3] (q^{2m+1}; q^{-2})_k; \quad (1.25)$$

for N even, we find

$$\begin{aligned} \frac{d\tilde{S}_{2m+2}}{d_q x} &= \sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} [2m+2-2k] (x \dot{-} 1)^{2m+1-2k} (q^{2m+1}; q^{-2})_k \\ &= [2m+2] \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (x \dot{-} 1)^{2m+1-2k} (q^{2m+1}; q^{-2})_k = [2m+2] \tilde{S}_{2m+1}, \end{aligned}$$

because

$$\begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} [2m+2-2k] = [2m+2] \begin{bmatrix} m \\ k \end{bmatrix}_{q^2}, \quad (1.26)$$

which is true in view of the obvious relation

$$[u]_{q^2} = [2u]_q / [2]_q. \quad (1.27)$$

It remains to verify formula (1.19), which is nothing but the Gauss formula (1.7). We shall verify the latter in 4 easy steps.

1st Step is formula (1.7a):

$$\begin{aligned} s_{2m+1|0} &= - \sum_{\ell \geq 0} \begin{bmatrix} 2m+1 \\ \ell \end{bmatrix} (-1)^\ell = \sum_{\ell \geq 0} \begin{bmatrix} 2m+1 \\ 2m+1-\ell \end{bmatrix} (-1)^{\ell-1} \\ &= \sum_{L \geq 0} \begin{bmatrix} 2m+1 \\ L \end{bmatrix} (-1)^{2m-L} = -s_{2m+1|0}, \end{aligned}$$

so that $s_{2m+1|0} = 0$;

2nd Step is formula (1.10b):

$$s_{2m|1} = \sum_{\ell \geq 0} \begin{bmatrix} 2m \\ \ell \end{bmatrix} (-q)^\ell = \sum_{\ell \geq 0} \begin{bmatrix} 2m \\ \ell \end{bmatrix} (-1)^\ell = s_{2m|0}. \quad (1.28)$$

Indeed,

$$\frac{s_{2m|1} - s_{2m|0}}{q-1} = \sum_{\ell \geq 0} \begin{bmatrix} 2m \\ \ell \end{bmatrix} (-1)^\ell [\ell] \text{ [by (1.22)]} = -[2m] \sum_{\ell \geq 1} \begin{bmatrix} 2m-1 \\ \ell-1 \end{bmatrix} (-1)^{\ell-1}$$

[by (1.7a)] = 0;

3rd Step is formula (1.10a):

$$\sum_{\ell \geq 0} \begin{bmatrix} 2m+1 \\ \ell \end{bmatrix} (-q)^\ell = (1 - q^{2m+1}) \sum_{\ell \geq 0} \begin{bmatrix} 2m \\ \ell \end{bmatrix} (-1)^\ell. \quad (1.29)$$

Indeed, since

$$\begin{bmatrix} 2m+1 \\ \ell \end{bmatrix} = \begin{bmatrix} 2m \\ \ell \end{bmatrix} + q^{2m+1-\ell} \begin{bmatrix} 2m \\ \ell-1 \end{bmatrix}, \quad (1.30)$$

we have:

$$\begin{aligned} \sum_{\ell \geq 0} \begin{bmatrix} 2m+1 \\ \ell \end{bmatrix} (-q)^\ell &= \sum_{\ell \geq 0} (-q)^\ell \begin{bmatrix} 2m \\ \ell \end{bmatrix} + \sum_{\ell \geq 1} (-q)^\ell q^{2m+1-\ell} \begin{bmatrix} 2m \\ \ell-1 \end{bmatrix} \text{ [by (1.28)]} \\ &= s_{2m|0} - q^{2m+1} \sum_{\ell \geq 1} \begin{bmatrix} 2m \\ \ell-1 \end{bmatrix} (-1)^{\ell-1} = (1 - q^{2m+1}) s_{2m|0}; \end{aligned}$$

4th Step is the last one: we prove that

$$s_{2m+2|0} = (1 - q^{2m+1}) s_{2m|0}, \quad (1.31)$$

from which the Gauss formula (1.76) follows at once, since

$$s_{2|0} = 1 - [2] + 1 = 1 - (1 + q) + 1 = 1 - q. \quad (1.32)$$

Now,

$$\begin{aligned} s_{2m+2|0} &= \sum_{\ell \geq 0} \begin{bmatrix} 2m+2 \\ \ell \end{bmatrix} (-1)^\ell \text{ [by (1.28)]} = \sum_{\ell \geq 0} \begin{bmatrix} 2m+2 \\ \ell \end{bmatrix} (-q)^\ell \text{ [by (1.30)]} \\ &= \sum_{\ell \geq 0} (-q)^\ell \begin{bmatrix} 2m+1 \\ \ell \end{bmatrix} + \sum_{\ell \geq 1} (-q)^\ell q^{2m+2-\ell} \begin{bmatrix} 2m+1 \\ \ell-1 \end{bmatrix} \text{ [by (1.29)]} \\ &= (1 - q^{2m+1}) s_{2m|0} - q^{2m+2} \sum_{\ell \geq 1} \begin{bmatrix} 2m+1 \\ \ell-1 \end{bmatrix} (-1)^{\ell-1} \text{ [by (1.7a)]} \\ &= (1 - q^{2m+1}) s_{2m|0}. \end{aligned}$$

We are done. Formula (1.16) is thereby proven. Substituting into this formula $x = 0$, we get an interesting identity

$$\sum_{k=0}^{\lfloor N/2 \rfloor} \begin{bmatrix} \lfloor N/2 \rfloor \\ k \end{bmatrix}_{q^2} q^{\binom{N-2k}{2}} (q^{N-\epsilon(N)}; q^{-2})_k = 1. \quad (1.33)$$

2 A different proof

To prove *polynomial* identities (1.12) generalizing Gauss' formulae (1.7), we had to prove independently the Gauss result along the way. This is not entirely agreeable. One ought to prove formulae (1.12) directly, by-passing the verification of the original Gauss formulae.

Such a proof follows.

Let $R_N(x)$ stand for either $S_N(x)$ or $\tilde{S}_N(x)$. We shall verify that

$$R_{N+1}(x) = xR_N(x) - R_N(qx). \quad (2.1)$$

Since

$$S_0(x) = \tilde{S}_0(x) = 1, \quad S_1(x) = \tilde{S}_1(x) = x - 1, \quad (2.2)$$

such a verification will prove that $S_N(x) = \tilde{S}_N(x)$ for all N .

We start with $R_N(x) = S_N(x)$. Let's look for a relation of the form

$$S_{N+1}(x) = BxS_N(bx) + AS_N(ax). \quad (2.3)$$

The x^0 - coefficients (recall that $S_n(x) = (-1)^n \sum_{\ell} \begin{bmatrix} n \\ \ell \end{bmatrix} (-x)^\ell$) yield

$$A = -1; \quad (2.4)$$

The x^{N+1} - coefficients yield

$$Bb^N = 1 \Leftrightarrow B = b^{-N}; \quad (2.5)$$

Finally, for $0 < r < N + 1$, the x^r - coefficients provide

$$\begin{bmatrix} N+1 \\ r \end{bmatrix} = Bb^{r-1} \begin{bmatrix} N \\ r-1 \end{bmatrix} + a^r \begin{bmatrix} N \\ r \end{bmatrix}. \quad (2.6)$$

In view of the relation (2.5), formula (2.6) can be rewritten as

$$\begin{bmatrix} N+1 \\ r \end{bmatrix} = (b^{-1})^{N+1-r} \begin{bmatrix} N \\ r-1 \end{bmatrix} + a^r \begin{bmatrix} N \\ r \end{bmatrix}. \quad (2.7)$$

Now, since

$$\begin{bmatrix} N+1 \\ r \end{bmatrix} = \begin{bmatrix} N \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} N \\ r \end{bmatrix} \quad (2.8a)$$

$$= q^{N+1-r} \begin{bmatrix} N \\ r-1 \end{bmatrix} + \begin{bmatrix} N \\ r \end{bmatrix}, \quad (2.8b)$$

equation (2.7) has two solutions:

$$b = 1, \quad a = q, \quad (2.9a)$$

$$b = q^{-1}, \quad a = 1. \quad (2.9b)$$

Thus,

$$S_{N+1}(x) = xS_N(x) - S_N(qx) \quad (2.10a)$$

$$= q^N x S_N(q^{-1}x) - S_N(x). \quad (2.10b)$$

(For $q = 1$, we get just *one* relation, $S_{N+1}(x) = (x-1)S_N(x)$.)

Denote by \mathcal{O} the linear operator acting on functions of x by the rule:

$$\mathcal{O}(f(x)) = xf(x) - f(qx). \quad (2.11)$$

We need to check that

$$\mathcal{O}(\tilde{S}_N) = \tilde{S}_{N+1}. \quad (2.12)$$

We shall check separately the cases of even and odd N :

$$\tilde{S}_{2m+1}(x) = \sum_{k=0}^m (x-1)^{2m+1-2k} c_{m|k}, \quad (2.13)$$

$$c_{m|k} = \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^{2m+1}; q^{-2})_k, \quad (2.14)$$

$$\tilde{S}_{2m}(x) = \sum_{k=0}^m (x-1)^{2m-2k} d_{m|k}, \quad (2.15)$$

$$d_{m|k} = \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^{2m-1}; q^{-2})_k. \quad (2.16)$$

To proceed further, let's establish first that

$$\mathcal{O}((x \dot{-} 1)^s) = (x \dot{-} 1)^{s+1} + q^{s-1}(1 - q^s)(x \dot{-} 1)^{s-1}. \quad (2.17)$$

Indeed,

$$\begin{aligned} \mathcal{O}((x \dot{-} 1)^s) &= x(x \dot{-} 1)^s - (qx \dot{-} 1)^s = ((x - q^s) + q^s)(x \dot{-} 1)^s - q^s(x \dot{-} 1q^{-1})^s \\ &= (x - q^s)(x \dot{-} 1)^s + q^s(x \dot{-} 1)^{s-1}(x - q^{s-1}) - q^s(x - q^{-1})(x \dot{-} 1)^{s-1} \\ &= (x \dot{-} 1)^{s+1} + q^s(x \dot{-} 1)^{s-1}((x - q^{s-1}) - (x - q^{-1})) \\ &= (x \dot{-} 1)^{s+1} + q^s(x \dot{-} 1)^{s-1}q^{-1}(1 - q^s). \end{aligned}$$

Now,

$$\begin{aligned} \mathcal{O}(\tilde{S}_{2m+1}) &= \sum_{k=0}^m c_{m|k} \mathcal{O}((x \dot{-} 1)^{2m+1-2k}) \\ &= \sum_{k=0}^m c_{m|k} ((x \dot{-} 1)^{2m+2-2k} + q^{2m-2k}(1 - q^{2m+1-2k})(x \dot{-} 1)^{2m-2k}) \\ &= \sum_{k=0}^{m+1} (x \dot{-} 1)^{2m+2-2k} (c_{m|k} + c_{m|k-1} q^{2m+2-2k}(1 - q^{2m+3-2k})), \end{aligned} \quad (2.18\ell)$$

while

$$\tilde{S}_{2m+2} = \sum_{k=0}^{m+1} (x \dot{-} 1)^{2m+2-2k} d_{m+1|k}, \quad (2.18r)$$

so we need to verify that

$$d_{m+1|k} = c_{m|k} + c_{m|k-1} q^{2m+2-2k}(1 - q^{2m+3-2k}), \quad (2.19)$$

which is

$$\begin{aligned} &= \begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} (q^{2m+1}; q^{-2})_k \\ &= \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^{2m+1}; q^{-2})_k + \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q^2} (q^{2m+1}; q^{-2})_{k-1} q^{2m+2-2k}(1 - q^{2m+3-2k}), \end{aligned} \quad (2.20)$$

which is equivalent to

$$\begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} = \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q^2} (1 - q^{2m+1-2(k-1)})^{-1} q^{2m+2-2k}(1 - q^{2m+3-2k}),$$

which is finally

$$\begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} = \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q^2} (q^2)^{m+1-k},$$

and this is so by formula (2.8b).

Next,

$$\begin{aligned}
 \mathcal{O}(\tilde{S}_{2m}) &= \sum_{k=0}^m d_{m|k} \mathcal{O}((x \dot{-} 1)^{2m-2k}) \\
 &= \sum_{k=0}^m d_{m|k} ((x \dot{-} 1)^{2m+1-2k} + q^{2m-2k-1} (1 - q^{2m-2k}) (x \dot{-} 1)^{2m-1-2k}) \\
 &= \sum_{k=0}^m (x \dot{-} 1)^{2m+1-2k} (d_{m|k} + d_{m|k-1} q^{2m+1-2k} (1 - q^{2m+2-2k})), \tag{2.21\ell}
 \end{aligned}$$

while

$$\tilde{S}_{2m+1} = \sum_{k=0}^m (x \dot{-} 1)^{2m+1-2k} c_{m|k}, \tag{2.21r}$$

so we need to check that

$$c_{m|k} = d_{m|k} + d_{m|k-1} q^{2m+1-2k} (1 - q^{2m+2-2k}), \tag{2.22}$$

which is

$$\begin{aligned}
 \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^{2m+1}; q^{-2})_k &= \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^{2m-1}; q^{-2})_k \\
 &+ \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q^2} (q^{2m-1}; q^{-2})_{k-1} q^{2m+1-2k} (1 - q^{2m+2-2k}), \tag{2.23}
 \end{aligned}$$

which is equivalent to

$$\begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (1 - q^{2m+1}) = \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (1 - q^{2m-1-2(k-1)}) + \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q^2} q^{2m+1-2k} (1 - q^{2m+2-2k}),$$

which can be rewritten as

$$\begin{bmatrix} m \\ k \end{bmatrix}_{q^2} q^{2m+1-2k} (1 - q^{2k}) = \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q^2} q^{2m+1-2k} (1 - q^{2m+2-2k}),$$

which is equivalent to

$$\begin{bmatrix} m \\ k \end{bmatrix} [k] = \begin{bmatrix} m \\ k-1 \end{bmatrix} [m+1-k],$$

which is obvious.

Remark 2.24. Set

$$\tilde{S}_N(x) = \sum_k e_{N|k} (x \dot{-} 1)^{N-2k}, \tag{2.25}$$

so that

$$c_{m|k} = e_{2m+1|k}, \quad d_{m|k} = e_{2m|k}. \quad (2.26)$$

Then the pair of equalities (2.19) and (2.22) can be rewritten as the single one:

$$e_{N+1|k} = e_{N|k} + e_{N|k-1} q^{N+1-2k} (1 - q^{N+2-2k}), \quad (2.27)$$

equivalent to the relation

$$\tilde{S}_{N+1} = \mathcal{O}(\tilde{S}_N).$$

3 The Taylor expansions point of view

Formula (1.16) (or (2.25)) is reminiscent of the Taylor expansion:

$$f(x) = \sum_{k \geq 0} \frac{f^{(k)}(a)}{k!} (x - a)^k, \quad (3.1)$$

where

$$f^{(i)}(x) = \left(\frac{d}{dx} \right)^i (f(x)). \quad (3.2)$$

There exist many different q -versions of the classical Taylor expansion. We shall make use below of the following particular one:

$$f(x) = \sum_{k \geq 0} \frac{f^{(k)}(a)}{[k]!} (x \dot{-} a)^k, \quad (3.3)$$

where now

$$f^{(k)}(x) = \left(\frac{d}{d_q x} \right)^k (f(x)). \quad (3.4)$$

We shall prove formula (3.3) for f being polynomial in x . It's enough to consider the case $f(x) = x^n$, so that

$$f^{(k)}(x) = [k]! \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k}, \quad (3.5)$$

and we thus have to check that

$$x^n = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} a^{n-k} (x \dot{-} a)^k. \quad (3.6)$$

This can be verified either directly, or deduced from the identity (formula (2.10) in [5], p. 75)

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} a^{n-k} (x \dot{+} b)^k = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} (a \dot{+} b)^k. \quad (3.7)$$

for $b = -a$.

Thus, formula (3.3) is proven. Taking $f(x)$ to be $S_N(x)$,

$$S_N(x) = (-1)^N \sum_k \begin{bmatrix} N \\ k \end{bmatrix} (-x)^k, \quad (3.8)$$

where, by formula (1.18a),

$$S_N^{(k)}(x) = [k]! \begin{bmatrix} N \\ k \end{bmatrix} S_{N-k}(x), \quad (3.9)$$

we get

$$S_N(x) = \sum_k \begin{bmatrix} N \\ k \end{bmatrix} G_{N-k}(x \dot{-} 1)^k = \sum_k \begin{bmatrix} N \\ k \end{bmatrix} (x \dot{-} 1)^{N-k} G_k, \quad (3.10)$$

where, by the Gauss formula (1.7),

$$G_k = S_k(1) = \begin{cases} 0, & k \text{ odd}, \\ (q^{k-1}; q^{-2})_{\lfloor k/2 \rfloor}, & k \text{ even}. \end{cases} \quad (3.11)$$

Thus,

$$S_N(x) = \sum_k \begin{bmatrix} N \\ 2k \end{bmatrix} (x \dot{-} 1)^{N-2k} (q^{2k-1}; q^{-2})_k. \quad (3.12)$$

Comparing formulae (1.12) and (3.12), we see that we must have

$$\begin{bmatrix} N \\ 2k \end{bmatrix}_q (q^{2k-1}; q^{-2})_k = \begin{cases} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^{2m+1}; q^{-2})_k, & N = 2m + 1 \\ \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^{2m-1}; q^{-2})_k, & N = 2m \end{cases} \quad (3.13)$$

and these relations can be easily verified. Thus,

$$(-1)^N \sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix} (-x)^k = \sum_{k=0}^{\lfloor N/2 \rfloor} \begin{bmatrix} N \\ 2k \end{bmatrix} (x \dot{-} 1)^{N-2k} (q^{2k-1}; q^{-2})_k. \quad (3.14)$$

Remark 3.15. Euler's formula (1.13) suggests that one should consider more general family of polynomials:

$$P_N(x) = \sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} x^\ell q^{\alpha \ell^2}, \quad (3.16)$$

with $\alpha = 0$ corresponding to the Gauss case, $\alpha = 1/2$ corresponding to the Euler case, and $\alpha = 1$ corresponding to the Szegő case [1,7]. Applying the arguments used above, we

find:

$$\frac{dP_N(x)}{d_q x} = [N]q^\alpha P_{N-1}(q^{2\alpha}x), \quad (3.17)$$

$$P_{N+1}(x) = q^\alpha x P_N(q^{2\alpha}x) + P_N(qx) \quad (3.18a)$$

$$= q^{N+\alpha} x P_N(q^{2\alpha-1}x) + P_N(x), \quad (3.18b)$$

$$P_N(x) = \sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix} \rho_{N-k}(x) \theta_k, \quad (3.19)$$

where

$$\rho_n(x) = q^{(1-2\alpha)\binom{n}{2}} (-q^{(2n-1)\alpha}x; q^{-1})_n \quad (3.20)$$

satisfies the same q -differential equation (3.17) as $P_n(x)$:

$$\frac{d\rho_n(x)}{d_q x} = [n]q^\alpha \rho_{n-1}(q^{2\alpha}x), \quad (3.21)$$

and θ_k 's are some x -independent connection coefficients. Unfortunately, I haven't been able to find a compact expression for the coefficients $\theta_k = \theta_k(q; \alpha)$.

4 The geometric progressions point of view

Formula (1.2)

$$\sum_{\ell=0}^N \binom{N}{\ell} (-1)^\ell = \delta_0^N, \quad N \in \mathbf{Z}_+, \quad (4.1)$$

can be equivalently put into the following interesting form:

$$\sum_{\ell=0}^{\infty} \frac{t^\ell}{(1+t)^{\ell+1}} = 1. \quad (4.2)$$

(We treat all series as formal power series, and so don't have to pay attention to questions of convergence. The series (4.2) converges for real $t > -1/2$.) Indeed, multiply equality (4.1) by $(-t)^N$ and then sum on all $N \in \mathbf{Z}_+$:

$$\begin{aligned} 1 &= \sum_{N,\ell} (-t)^N \binom{N}{\ell} (-1)^\ell = \sum_{s,\ell} (-t)^{s+\ell} \binom{s+\ell}{\ell} (-1)^\ell = \sum_{\ell \geq 0} t^\ell \sum_{s \geq 0} \binom{s+\ell}{\ell} (-t)^s \\ &= \sum_{\ell \geq 0} \frac{t^\ell}{(1+t)^{\ell+1}}, \end{aligned}$$

where we used the following version of the Newton's binomial

$$\frac{1}{(1-t)^{N+1}} = \sum_{s \geq 0} \binom{N+s}{s} t^s. \quad (4.3)$$

We can perform similar conversion upon the formula (1.5), an Euler-type q -analogue of formula (4.1). Multiply the equality

$$\sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (-1)^\ell q^{\binom{\ell}{2}} = \delta_0^N, \quad N \in \mathbf{Z}_+, \quad (4.4)$$

by $(-t)^N$ and sum over all $N \in \mathbf{Z}_+$:

$$\begin{aligned} 1 &= \sum_{N, \ell} (-t)^N \begin{bmatrix} N \\ \ell \end{bmatrix} (-1)^\ell q^{\binom{\ell}{2}} = \sum_{s, \ell \geq 0} (-t)^{s+\ell} \begin{bmatrix} s+\ell \\ \ell \end{bmatrix} (-1)^\ell q^{\binom{\ell}{2}} \\ &= \sum_{\ell} t^\ell q^{\binom{\ell}{2}} \sum_s \begin{bmatrix} s+\ell \\ \ell \end{bmatrix} (-t)^s \text{ [by (4.6)]} = \sum_{\ell \geq 0} \frac{t^\ell q^{\binom{\ell}{2}}}{(1+t)^{\ell+1}}. \end{aligned}$$

Thus,

$$\sum_{\ell=0}^{\infty} \frac{t^\ell q^{\binom{\ell}{2}}}{(1+t)^{\ell+1}} = 1; \quad (4.5)$$

we used in the calculation above the following Euler version of formula (4.3):

$$\frac{1}{(1+t)^{N+1}} = \sum_{s \geq 0} \begin{bmatrix} N+s \\ s \end{bmatrix} t^s. \quad (4.6)$$

Let us now apply the same conversion device to the Gauss result (1.7):

$$G_N = \sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix} (-1)^k = \begin{cases} 0, & N \text{ odd,} \\ (q^{N-1}; q^{-2})_{\lfloor N/2 \rfloor}, & N \text{ even.} \end{cases} \quad (4.7)$$

Multiplying by $(-t)^N$ and summing on N we find:

$$\begin{aligned} \sum_N (-t)^N G_N &= \sum_m t^{2m} (q^{2m-1}; q^{-2})_m = 1 + \sum_{m=1}^{\infty} (1-q) \dots (1-q^{2m-1}) t^{2m} \\ &= \sum_N (-t)^N \sum_k \begin{bmatrix} N \\ k \end{bmatrix} (-1)^k = \sum_{s, k} (-t)^{k+s} \begin{bmatrix} k+s \\ k \end{bmatrix} (-1)^k \\ &= \sum_k t^k \sum_s \begin{bmatrix} k+s \\ k \end{bmatrix} (-t)^s = \sum_{k \geq 0} \frac{t^k}{(1+t)^{k+1}}. \end{aligned}$$

Thus,

$$\sum_{k \geq 0} \frac{t^k}{(1+t)^{k+1}} = 1 + \sum_{m=1}^{\infty} (1-q) \dots (1-q^{2m-1}) t^{2m}. \quad (4.8)$$

This formula is the first from a pair found by Carlitz in [3]. The second formula in that pair is the case $\{r=1\}$ of the following general relation

$$\sum_{\ell=0}^{\infty} \frac{(q^r t)^\ell q^{\binom{\ell}{2}}}{(1+t)^{\ell+1}} = \sum_{N \geq 0} (1-q^r)^N (-t)^N, \quad (4.9)$$

which can be proven as follows:

$$\begin{aligned} \sum_{\ell=0}^{\infty} \frac{(q^r t)^\ell q^{\binom{\ell}{2}}}{(1+t)^{\ell+1}} &= \sum_{\ell} q^{r\ell} t^\ell q^{\binom{\ell}{2}} \sum_s \begin{bmatrix} \ell+s \\ \ell \end{bmatrix} (-t)^s = \sum_{N \geq 0} t^N \sum_{\ell=0}^N (-1)^{N-\ell} \begin{bmatrix} N \\ \ell \end{bmatrix} q^{\binom{\ell}{2}} (q^r)^\ell \\ &= \sum_N (-t)^N \sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} q^{\binom{\ell}{2}} (-q^r)^\ell \text{ [by (1.3)]} = \sum_N (-t)^N (1+q^r)^N. \end{aligned}$$

For $r = 0$, formula (4.9) becomes formula (4.5). Since r is arbitrary, replacing in formula (4.9) tq^r by another variable z , we get

$$\sum_{\ell=0}^{\infty} \frac{z^\ell q^{\binom{\ell}{2}}}{(1+t)^{\ell+1}} = \sum_{N \geq 0} (-1)^N (t+qz)^N, \quad (4.10)$$

a q -analogue of the geometric progression formula

$$\frac{1}{1+t} \sum_{\ell=0}^{\infty} \left(\frac{z}{1+t} \right)^\ell = \sum_{N=0}^{\infty} (z-t)^N. \quad (4.11)$$

5 Gauss-like non-alternating sums

For $x = -1$, Newton's formula (1.1) yields

$$\sum_{\ell=0}^N \binom{N}{\ell} = 2^N. \quad (5.1)$$

Similarly, the Euler binomial (1.3) for $x = -q$ provides

$$\sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} q^{\binom{\ell+1}{2}} = (1+q)^N. \quad (5.2)$$

If we apply to these two banalities Gauss-like ansatz, we should look at the sums of the form

$$\sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (q^r)^\ell. \quad (5.3)$$

Not much is known about such sums, at least as far as I can tell. (See Remark 6.12.) However, we shall see below that for $r = 1/2$,

$$\sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} q^{\ell/2} = (-q^{1/2}; q^{1/2})_N. \quad (5.4)$$

Changing q into q^2 , this formula may be rewritten in the form

$$\sigma_N = \sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix}_{q^2} q^\ell = (1+q)^N. \quad (5.5)$$

Let's prove it. This formula is obviously true for $N = 0, 1$. Using induction on N and observing that

$$\sigma_N = \sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix}_{q^2} q^\ell = \sum_{\ell=0}^N \begin{bmatrix} N \\ N-\ell \end{bmatrix}_{q^2} q^\ell = \sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix}_{q^2} q^{N-\ell}, \quad (5.6)$$

we find:

$$\begin{aligned} \sigma_{N+1} &= \sum_{\ell \geq 0} \begin{bmatrix} N+1 \\ \ell \end{bmatrix}_{q^2} q^\ell \text{ [by (2.8b)]} = \sum_{\ell=0}^N \left(\begin{bmatrix} N \\ \ell \end{bmatrix}_{q^2} + \begin{bmatrix} N \\ \ell-1 \end{bmatrix}_{q^2} q^{2N+2-2\ell} \right) q^\ell \\ &= \sigma_N + \sum_{\ell \geq 0} \begin{bmatrix} N \\ \ell \end{bmatrix}_{q^2} q^{2N+1-\ell} \text{ [by (5.6)]} = \sigma_N + q^{N+1} \sigma_N = (1 + q^{N+1}) \sigma_N. \end{aligned} \quad (5.7)$$

Thus,

$$\sigma_{N+1} = (1 + q^{N+1}) \sigma_N, \quad (5.8)$$

and since $\sigma_0 = 1$, formula (5.5) follows.

The *derivation* of formula (5.7) above suggests consideration of more general sums

$$\sigma_N(\gamma) = \sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} q^{\gamma k}. \quad (5.9)$$

Since

$$\sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} q^{\gamma k} = \sum_{k=0}^N \begin{bmatrix} N \\ N-k \end{bmatrix}_{q^2} q^{\gamma k} = \sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} q^{\gamma(N-k)} = q^{\gamma N} \sigma_N(-\gamma),$$

we find that

$$\sigma_N(-\gamma) = q^{-\gamma N} \sigma_N(\gamma). \quad (5.10)$$

Further,

$$\begin{aligned} \sigma_{N+1}(\gamma) &= \sum_{k=0}^N \begin{bmatrix} N+1 \\ k \end{bmatrix}_{q^2} q^{\gamma k} \text{ [by (2.8a)]} = \sum_{k=0}^N (q^{2k} \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} N \\ k-1 \end{bmatrix}_{q^2}) q^{\gamma k} \\ &= \sigma_N(\gamma + 2) + q^\gamma \sigma_N(\gamma), \end{aligned}$$

so that

$$\sigma_N(\gamma + 2) = \sigma_{N+1}(\gamma) - q^\gamma \sigma_N(\gamma). \quad (5.11)$$

Since we have already calculated $\sigma_N = \sigma_N(1)$ (5.5), formula (5.11) allows us to find $\sigma_N(\gamma)$ for arbitrary odd γ .

Setting

$$\sigma_N(2\ell + 1) = \sigma_N(1) \sum_{s=0}^{\ell} c_{\ell|s} q^{\binom{s+1}{2}} Q^s, \quad Q = q^N, \quad \ell \in \mathbf{Z}_+, \quad (5.12)$$

we can translate the recurrence relation (5.11) into the form

$$c_{\ell+1|s} = (q^s - q^{2\ell+1})c_{\ell|s} + c_{\ell|s-1}, \quad (5.13)$$

with the understanding that

$$c_{\ell|s} = 0 \quad \text{unless} \quad 0 \leq s \leq \ell. \quad (5.14)$$

Since

$$c_{0|0} = 1, \quad (5.15)$$

a little calculation shows that

$$c_{\ell|2r} = \begin{bmatrix} \ell - r \\ r \end{bmatrix}_{q^2} \frac{g_{\ell-r}}{g_r}, \quad (5.16a)$$

$$c_{\ell|2r+1} = \begin{bmatrix} \ell - r - 1 \\ r \end{bmatrix}_{q^2} \frac{g_{\ell-r}}{g_{r+1}}, \quad (5.16b)$$

where g_i 's are the Gauss products:

$$g_i = \prod_{t \text{ odd} < 2i} (1 - q^t), \quad i \in \mathbf{N}; \quad g_0 = 1. \quad (5.17)$$

It's easy to verify that formulae (5.16) satisfy the recurrence relation (5.13) and the boundary condition (5.15). It's interesting to observe that formula (5.16) exhibits still another form of 2-periodicity.

The first few $\sigma_N(2\ell + 1)$'s are written below:

$$\sigma_N(3)/\sigma_N(1) = (1 - q) + qQ, \quad (5.18a)$$

$$\sigma_N(5)/\sigma_N(1) = (1 - q)(1 - q^3) + qQ(1 - q^3) + q^3Q^2, \quad (5.18b)$$

$$\begin{aligned} \sigma_N(7)/\sigma_N(1) &= (1 - q)(1 - q^3)(1 - q^5) + qQ(1 - q^3)(1 - q^5) \\ &\quad + q^3Q^2(1 - q^3)[2]_{q^2} + q^6Q^3, \end{aligned} \quad (5.18c)$$

$$\begin{aligned} \sigma_N(9)/\sigma_N(1) &= (1 - q)(1 - q^3)(1 - q^5)(1 - q^7) + qQ(1 - q^3)(1 - q^5)(1 - q^7) \\ &\quad + q^3Q^2(1 - q^3)(1 - q^5)[3]_{q^2} + q^6Q^3(1 - q^5)[2]_{q^2} + q^{10}Q^4. \end{aligned} \quad (5.18d)$$

Passing to the limit $N \rightarrow \infty$ and considering $|q| < 1$, so that $Q = q^N \rightarrow 0$, we find:

$$\lim_{N \rightarrow \infty} \sigma_N(2\ell + 1)/\sigma_N(1) = (1 - q)(1 - q^3) \dots (1 - q^{2\ell-1}), \quad \ell \in \mathbf{N}. \quad (5.19)$$

Since

$$\sigma_\infty(\gamma) = \lim_{N \rightarrow \infty} \sigma_N(\gamma) = \sum_{k \geq 0} \begin{bmatrix} \infty \\ k \end{bmatrix}_{q^2} q^{\gamma k} = 1 + \sum_{k > 0} \frac{q^{\gamma k}}{(1 - q^2) \dots (1 - q^{2k})}, \quad (5.20)$$

formula (5.19) can be rewritten as

$$\sum_{k \geq 0} \frac{q^{(2\ell+1)k}}{(q^2; q^2)_k} = (q; q^2)_\ell \sum_{k \geq 0} \frac{q^k}{(q^2; q^2)_k}. \quad (5.21)$$

Now

$$(a; \rho)_\ell = (a; \rho)_\infty / (\rho^\ell a; \rho)_\infty, \quad (5.22)$$

so that formula (5.21) can be rewritten as

$$\frac{1}{(q; q^2)_\infty} \sum_{k=0}^{\infty} \frac{z^k}{(q^2; q^2)_k} = \frac{1}{(z; q^2)_\infty} \sum_{k=0}^{\infty} \frac{q^k}{(q^2; q^2)_k}, \quad (5.23)$$

where we introduced

$$z = q^{2\ell+1}. \quad (5.24)$$

Formula (5.23) is true as it stands, for *arbitrary* z , because the difference of the LHS and the RHS of this formula is an analytic function of z for $|z| < 1$, vanishing for an infinite number of different values $z = q^{2\ell+1}$, $\ell \in \mathbf{Z}_+$, condensing to zero.

Remark 5.25. The *alternating* Gauss-like sums (1.9)

$$(-1)^N s_{N|r} = \sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (-1)^\ell (q^r)^\ell \quad (5.26)$$

have been effectively calculated in Section 1 for *integer* $r \in \mathbf{Z}$. The *non-alternating* sums (5.3)

$$\sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (q^r)^\ell \quad (5.27)$$

have been effectively calculated in this section for *half-integers* $r \in \frac{1}{2} + \mathbf{Z}$. There must be some underlying reasons for this dichotomy.

6 Remarks

Remark 6.1. The basic philosophy of q -language is *multiplicative* discretization of classical continuous mathematics. Interestingly enough, the formulae in this paper can be interpreted as statements in an *additive* discrete language, a certain q -analogue of the classical difference calculus. The latter can be summarized as follows.

Let $\boldsymbol{\theta} = (\theta(0), \theta(1), \dots)$ be a fixed sequence. For every sequence $\{a_n\}$, define the q -difference sequences

$$(\Delta^0 a)_n = a_n, \quad (6.1a)$$

$$(\Delta^{k+1} a)_n = (\Delta^k a)_{n+1} - q^{\theta(k)} (\Delta^k a)_n, \quad k \in \mathbf{Z}_+. \quad (6.1b)$$

When the parameter $\boldsymbol{\theta}$ has the canonical form

$$\theta(k) = k, \quad k \in \mathbf{Z}_+, \quad (6.2)$$

the sequences $\{(\Delta^k a)_n | k, n \in \mathbf{Z}_+\}$ can be reconstructed from the boundary conditions

$$b_k = (\Delta^k a)_0, \quad k \in \mathbf{Z}_+, \quad (6.3)$$

by the easily verifiable formula

$$(\Delta^k a)_n = \sum_{s=0}^n b_{k+n-s} \begin{bmatrix} n \\ s \end{bmatrix} q^{ks}. \quad (6.4)$$

In particular, when $k = 0$ we get

$$a_n = (\Delta^0 a)_n = \sum_{s=0}^n b_{n-s} \begin{bmatrix} n \\ s \end{bmatrix} = \sum_{s=0}^n b_s \begin{bmatrix} n \\ s \end{bmatrix}. \quad (6.5)$$

Thus, evaluation of the sums (5.26) and (5.27):

$$\sum_{\ell=0}^N \begin{bmatrix} N \\ \ell \end{bmatrix} (\pm q^r)^\ell, \quad (6.6)$$

can be thought of as the process of reconstruction of the original sequence $\{a_N\}$ given the boundary q -difference sequence $\{(\Delta^n a)_0 = (\pm q^r)^n\}$.

In a superficially more general direction, say for the nonalternating case, if we fix $r, \rho \in \mathbf{Z}_+$ and set

$$b_s = \begin{bmatrix} s \\ \rho \end{bmatrix} q^{\alpha(s)}, \quad \alpha(s) = (s - \rho)(r + \frac{1}{2}), \quad (6.7)$$

we find

$$\begin{aligned} a_n &= \sum_{s=0}^n \begin{bmatrix} n \\ s \end{bmatrix} b_s = \sum_{s=0}^n \begin{bmatrix} n \\ s \end{bmatrix} \begin{bmatrix} s \\ \rho \end{bmatrix} q^{\alpha(s)} = \begin{bmatrix} n \\ \rho \end{bmatrix} \sum_{s=\rho}^n \begin{bmatrix} n - \rho \\ s - \rho \end{bmatrix} q^{\alpha(s)} \\ &= \begin{bmatrix} n \\ \rho \end{bmatrix} \sum_{s=0}^{n-\rho} \begin{bmatrix} n - \rho \\ s \end{bmatrix} q^{s(r+\frac{1}{2})} = \begin{bmatrix} n \\ \rho \end{bmatrix} \tilde{\sigma}_{n-\rho}(2r+1), \end{aligned} \quad (6.8)$$

where

$$\tilde{\sigma}_N(\gamma; q) = \sigma_N(\gamma; q^{\frac{1}{2}}). \quad (6.9)$$

In particular, for $r = 0$ and $\rho = 1$, formula (6.8) yields:

$$a_n = [n](-q^{\frac{1}{2}}; q^{\frac{1}{2}})_{n-1}. \quad (6.10)$$

When $q = 1$, this becomes S. Rabinowitz's Crux 946 formula ([6], p. 194)

$$a_n = n \cdot 2^{n-1}, \quad b_n = n, \quad n \in \mathbf{Z}_+. \quad (6.11)$$

Remark 6.12. Many formulae in this paper can be found in the literature. The polynomials $(-1)^N S_N(-x)$ (1.11) are called by Andrews "Rogers-Szegő polynomials", and many of their interesting properties are listed on pp. 49-51 in [2]. Andrews also provides a very short proof of the Gauss formulae (1.7), on p. 37 in [2]. N. J. Fine has also studied these polynomials; formula (5.5) can be found on p. 29 of his book [4], as well as on p. 49 of the Andrews book [2].

Remark 6.13. The Gauss device can be thought of as chopping off the naturally occurring factors $q^{\binom{n}{2}}$ from the Euler q -analogue (1.32) of Newton's binomial (1.1). In the opposite spirit, one can ask about what happens when we *attach* these factors to a place that is naturally missing them, another Euler's form of Newton's binomial, formula (4.6):

$$V_N(t) = \sum_{s \geq 0} \begin{bmatrix} N+s \\ s \end{bmatrix} t^s q^{\binom{s}{2}}. \quad (6.14)$$

Since these objects are no longer polynomials but are in fact infinite series, we won't pursue this avenue here and leave it to the reader as an exercise. The numbers $v_N = V_N(q)$ can be found on p. 8 of Fine's book [4]:

$$v_{2k} = \frac{1}{(q^2; q^2)_k} \sum_{n \geq 0} q^{\binom{n+1}{2}} = \frac{1}{(q^2; q^2)_k} \prod_{n \geq 1} \left(\frac{1 - q^{2n}}{1 - q^{2n-1}} \right), \quad (6.15a)$$

$$v_{2k+1} = \frac{1}{(q; q^2)_k} = \frac{1}{(1-q)(1-q^3)\dots(1-q^{2k+1})}. \quad (6.15b)$$

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